Sonar Equation:
The Wave Equation

The Great Wave

Hokusai

LO: Recognize physical principles associated with terms in sonar equation.

…the Punchline

If density too high to resolve individual organisms, then:

E[energy from volume] = n * energy from an individual

energy from individual = s_{bs}= backscattering cross section

energy from volume = s_v = volume backscatter coefficient

$$E[s_v] = n\ s_{bs}$$

So…

- measure energy from volume

- assume, measure, or model energy from representative individual

- can calculate density of individuals

Definitions

Wave Number:

$k = 2\pi/\lambda$ (units rad/m)

Angular Frequency:

$w = 2\pi/T=2\pi f$ (units rad/s)

Pressure (p): force/area (units Pascals = N/m^2, $1 mPa = 10^{-6}$ Pa)

$$p(r) = p(r_o)\, r_o/r$$

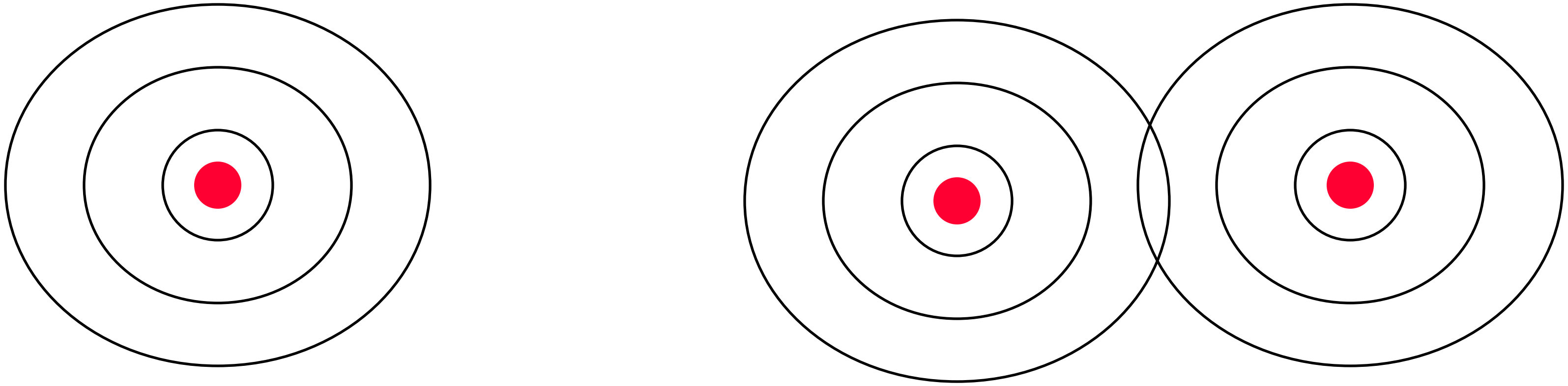
Intensity (I): power/area (units W/m^2)

$$I(r) = I(r_o)\,(r_o/r)^2$$

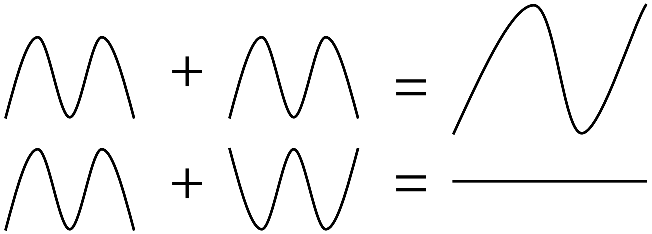
$$I \propto p^2$$

Hugen’s Principle

Every point in a wave field acts as a point source



Interference among sources (e.g. ripples in a pond)



Constructive

Destructive

Propagating Waves

speed 2 ms^{-1}

$D_2(x) = D_1(x\text{-}10) = D_1(x - 2\ ms^{-1} * 5\ s)$

$D(x) = D_1(x\text{-}ct)$

T = period

T = time

Since $c = \lambda/T$

$D(x) = D_1(x\text{-}\lambda/T * t) = D_1(x\text{-}\lambda ft)$

Make non-dimensional: divide by λ

$$D(x/\lambda) = D_1(x/\lambda - ft)$$

Non-dimensional Wave Equation

Make non-dimensional: divide by λ

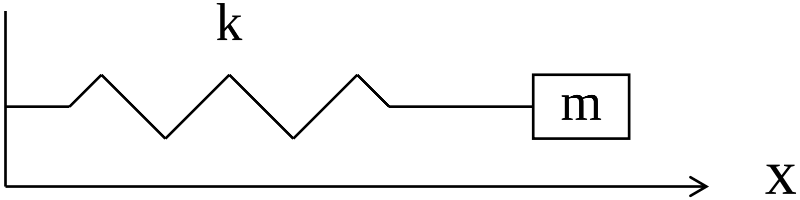
$D(x/\lambda) = D_1(x/\lambda - ft)$

Recall $2\pi/\lambda = k$; $2\pi f = \omega$

$D(k,x) = D1(kx - \omega t)$ +x direction

$D(k,x) = D1(kx + \omega t)$ -x direction

Harmonic Motion



Force = -kx

F = ma

Therefore: ma = -kx

$$m\frac{d^2x}{dt^2} = -kx \qquad 0 = -kx - m\frac{d^2x}{dt^2}$$

Divide by –m:

$$0 = \frac{d^2x}{dt^2} + \frac{kx}{m}$$

Harmonic Motion

x made up of 2 parts: $x_1 = A\cos\left(\sqrt{\frac{k}{m}}t\right)$ $x_2 = B\sin\left(\sqrt{\frac{k}{m}}t\right)$

k Units

k = N/kg

= (kg/s^2)/kg

= s^{-2}

Square root = s^{-1}

= rad/s

= ω

So: $x = A\cos\left(\sqrt{\frac{k}{m}}t\right) + B\sin\left(\sqrt{\frac{k}{m}}t\right)$ since $\omega = \sqrt{\frac{k}{m}}$

$$x = A\cos(\omega t) + B\sin(\omega t)$$

Sinusoidal form with magnitude $C = \sqrt{A^2 + B^2}$ and phase $\alpha = \tan^{-1}\left(\frac{B}{A}\right)$

$$x = C\cos(\omega t - \alpha)$$

A,B,C complex constants

Alternate Solution for Harmonic Motion

x made up of 2 parts: $x_1 = D e^{i\omega t}$ $x_2 = E e^{-i\omega t}$

So: $x = D e^{i\omega t} + E e^{-i\omega t}$

Wave Equation for Acoustics

$$\nabla^2 p = \frac{\delta^2 p}{\delta^2 t^2} * \frac{1}{c^2}$$

For plane waves in 1 direction: $\nabla^2 = \frac{\delta^2}{\delta x^2}$

LaPlacian operator

$$\frac{\delta^2 p}{\delta x^2} = \frac{\delta^2 p}{\delta^2 t^2} * \frac{1}{c^2}$$

Plane Wave Incident Solution

$$p(x,t) = Ae^{i(kx-\omega t)} + Be^{i(kx+\omega t)}$$

forward reverse

Forward Traveling Planar Wave:

p(x,t)= C cos(kx-ωt), where C is reference pressure (p_o)

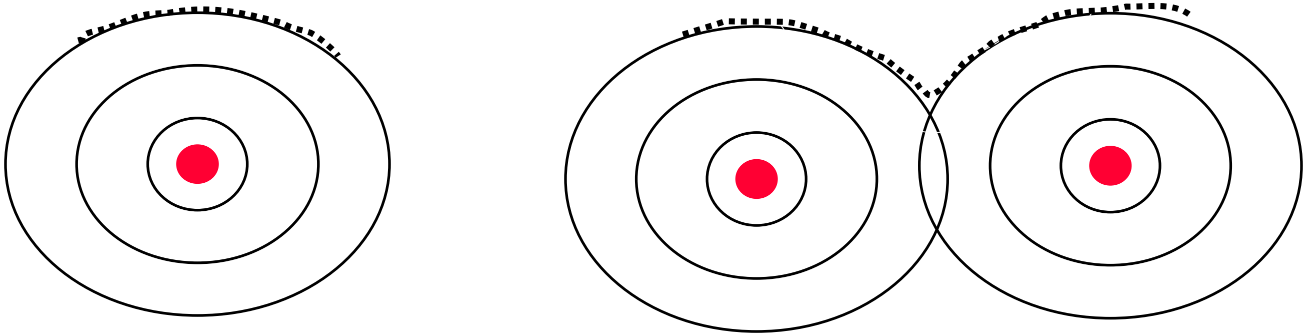
Alternate form:

$$p = \frac{p_o r_o}{r} e^{i(kr-\omega t)}$$

r = x = range

Planar Incident Pressure

$$p_{inc} = p_o r_o e^{\frac{i(kr_t - \omega t)}{r_t}} = \frac{p_o r_o}{r_t} e^{i(kr - \omega t)}$$



Spherical Scattering

LaPlacian $\nabla^2 = \frac{\delta^2}{\delta r^2} + \frac{2}{r}\frac{\delta}{\delta r}$

Wave Equation $\frac{\delta^2 p}{\delta r^2} + \frac{2}{r}\frac{\delta p}{\delta r} = \frac{1}{c^2}\frac{\delta^2 p}{\delta t^2}$

r is distance from source center

Solution $p = \frac{A}{r}e^{i(kr-\omega t)} = \frac{p_o r_o}{r}e^{i(kr-\omega t)}$

amplitude

frequency

Transducer Directivity

$$D(\theta) = \frac{\sin\left(\frac{kL}{2}\sin\theta\right)}{\frac{k}{L}\sin\theta} = \text{sinc}\left(\frac{kL}{2}\sin\theta\right)$$

$\text{sinc}=(\sin(x)/x)$

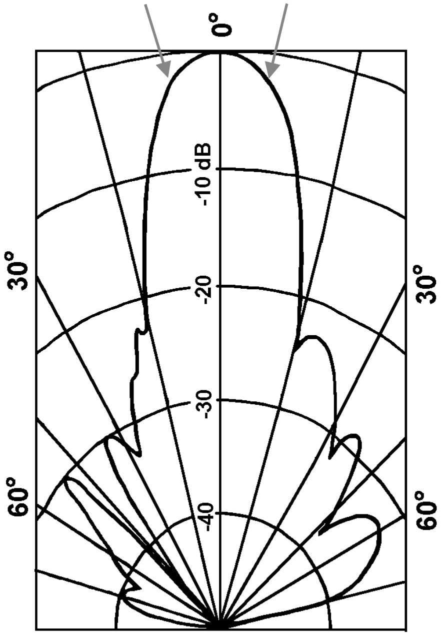
Directivity Index

$D_i = 10\log(D) = 10\log(I_o/\bar{I})$

where:

I_o = radiated intensity at acoustic axis

$\bar{I}$ = mean intensity over all directions



Calculate from transducer

$D_i = 10\log(4\pi a/\lambda)$ where a = active transducer area

Calculate from beam angles

$D_i = 10\log(2.5/\sin(\beta_1/2)\sin(\beta_2/2))$

where β's = beam width at –3dB points

Don’t Forget Transmission Losses

Spherical Spreading

$\Delta p \, \alpha \, 1/r$

$I \propto p^2$

$\Delta I \, \alpha \, 1/r^2$

Absorption

$$p_2 = p_1 \, 10^{- \alpha r/20}$$

$$\alpha = 1/r_o \, 10 \log_{10}(p_1{}^2/p_2{}^2)$$

$$I_2 = I_1 \, 10^{- \alpha r/10}$$

r_o = unit distance

$$\alpha = 1/r_o \, 10 \log_{10}(I_1/I_2)$$

Putting Components Together

Pressure

$$p(r) = p(r_o)\, r_o/r\ 10^{-\ \alpha(r\text{-}r_o/20)}$$

Intensity

$$I(r) = I(r_o)\, (r_o/r)^2\ 10^{-\ \alpha(r\text{-}r_o/10)}$$

Sonar Equation for Pressure

$$p(r) = p(r_o)\ r_o/r\ 10^{- \alpha(r-r_o/20)}$$

Divide by reference pressure p_o

$$p(r)/p_o = p(r_o)\ /p_o\ r_o/r\ 10^{- \alpha(r-r_o/20)}$$

Take 20 $\log_{10}$ of both sides:

$$20 \log_{10}(p(r)/p_o) = 20 \log_{10} (p(r_o) /p_o) + 20 \log_{10}(r_o/r) - \alpha(r-r_o)$$

Sonar Equation for Pressure

$$20 \log_{10}(p(r)/p_o) = 20 \log_{10} (p(r_o) /p_o) - 20 \log_{10}(r_o/r) - \alpha(r\text{-}r_o)$$

SPL = SL - TL - TL

Sound Pressure Level Source Level Transmission Losses

Echo Level (one way)

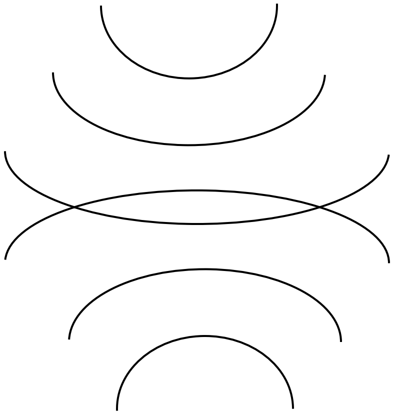
Scattering from an Object

I_i

Incident sound on target

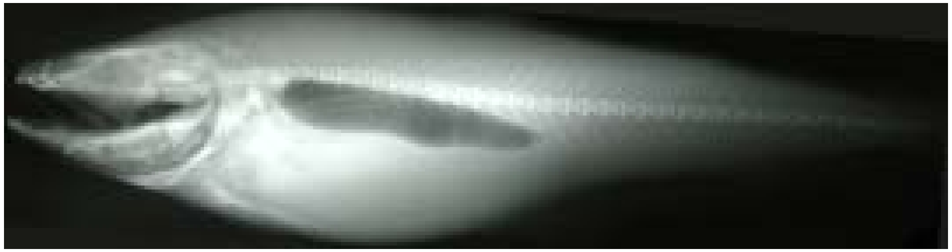
I_r

Intensity of reflected sound at 1 meter



$\rho_1 c_1$

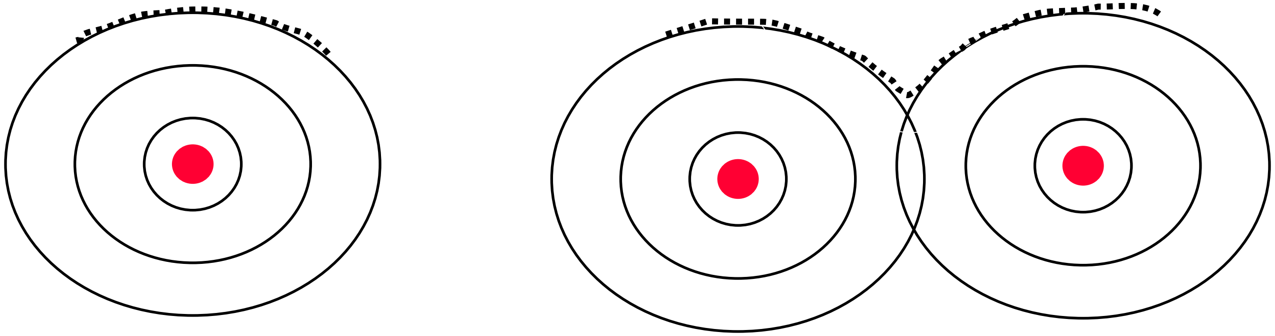
$\rho_2 c_2$



$$TS = 10\log_{10}\left(\frac{I_r}{I_i}\right)$$

Point or Spherical Scattering

Scattered Field



$$p_{scat} = \frac{p_o r_o}{r_t} e^{i\left(kr_t - \omega t\right)} \times \frac{e^{ikr_s}}{r_s} \times f$$

f = complex scattering amplitude

(a.k.a. complex scattering length)

amplitude frequency

reflectivity, target size, orientation, geometry of transmit & receive

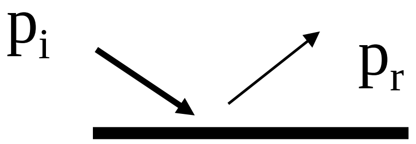
incident field

reflected field

Acoustic Impedance

Scattering is caused by an Acoustic Impedance mismatch

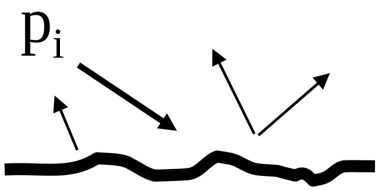
Reflection (1 direction)



LARGE objects (e.g. breakwater)

$R = p_r/p_i$

Scattering (all directions)



small objects (e.g. piling)

What is acoustic impedance (Z)? $Z = \rho\ c$

$g = \rho_2/\rho_1$

density

$h = c_2/c_1$

sound speed

Echoes: Acoustic Impedance Mismatch

Reflection

g = ρ_2/ρ_1

density contrast

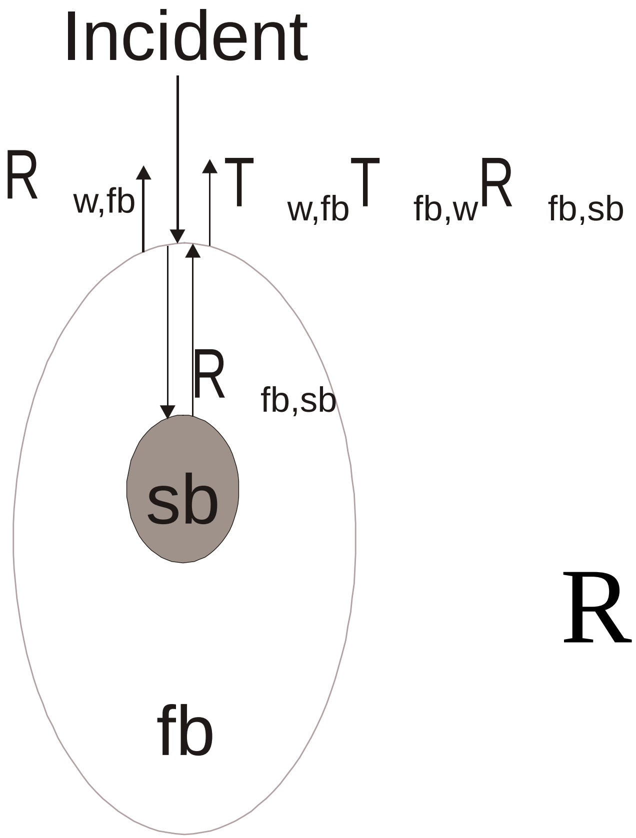
h = c_2/c_1

sound speed contrast

$$R = \frac{\rho_2 c_2 - \rho_1 c_1}{\rho_2 c_2 + \rho_1 c_1} = \frac{\frac{\rho_2 c_2}{\rho_1 c_1} - 1}{\frac{\rho_2 c_2}{\rho_1 c_1} + 1} = \frac{gh - 1}{gh + 1}$$

Reflection Coefficient

proportion of sound reflected at an interface



R = reflected
T = transmit
w = water
fb = fish body
sb = swim bladder

$$\mathrm{R}_{fb,sb} = \frac{g_{fb,sb} h_{fb,sb} - 1}{g_{fb,sb} h_{fb,sb} + 1}$$

Reflectivity Examples

Lead in water

$g = 11.35$

$h = 1.49$

$R = 14/16 \cong 1$

Perfect reflector

Air – water interface

$g = 0.0012$

$h = 0.22$

$R = 0-1/0+1 = -1$

Pressure release surface

Artic krill (*Euphasia superba*)

$g = 1.0357$

$h = 1.0279$

$R = 0.06/2.06 \cong 0.03$

Atlantic cod (*Gadus morhua*)

Body	Swimbladder
$g = 1.039$	$g = 0.0012$
$h = 1.054$	$h = 0.232$
$R = 0.045$	$R = -0.99$

Backscatter Definition

f = complex scattering amplitude (a.k.a. complex scattering length)

- if f is big then target scatters lots of sound
- backscatter is most common geometry for a transducer

σ_{bs} – backscattering cross-sectional area (units m^2)

TS – target strength (units dB re 1 μPa)

$$\sigma_{bs} = f_{bs} \times f_{bs}{}^{*} = \left| f_{bs} \right|^2$$

* complex conjugate

$$TS = 10\log_{10}\left(\left| f_{bs} \right|^2\right) = 10\log_{10}\left(\sigma_{bs}\right)$$

Some Final Definitions

EL = echo level $= 10\log_{10}\left(\frac{\left|p_r\right|^2}{(1\mu Pa)^2}\right)$

SL = source level $= 10\log_{10}\left(\frac{{p_i}^2{r_i}^2}{(1\mu Pa)^2(1m)^2}\right)$

TL = transmission loss $= 10\log_{10}\left(\frac{{r_{\text{target}}}^2}{(1m)^2}\right)$ assumes no absorption

TS = target strength $= 10\log_{10}\left(\frac{\sigma_{bs}}{(1m)^2}\right)$

Two Way Scattering Equation

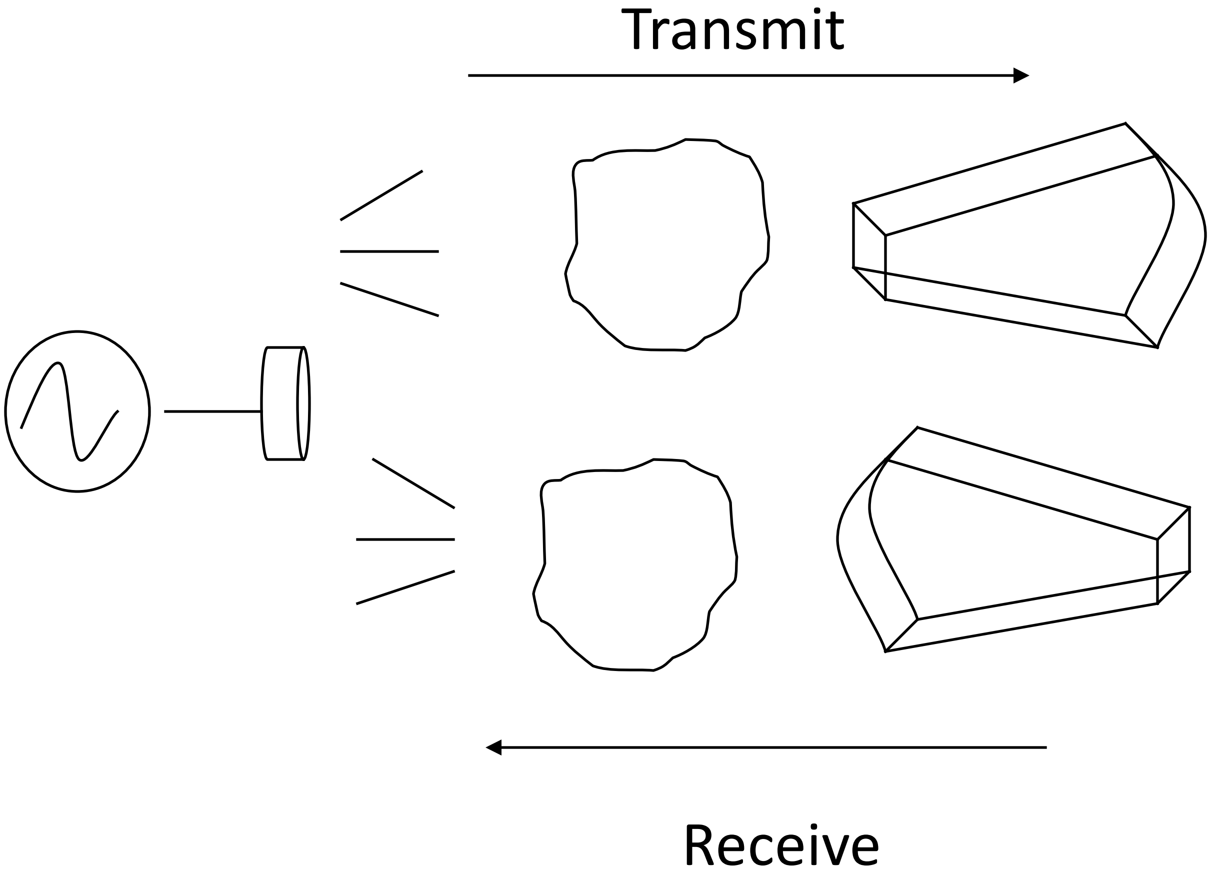
$$\left|p_{scat}\right|^2 = \left(p_o r_o\right)^2 \times \frac{1}{r_{\text{target}}{}^2} \times \frac{1}{r_{source}{}^2} \times \sigma_{bs}$$

$$\text{EL} \;=\; \text{SL} \;-\; \text{TL}_{\text{to target}} + \;\text{TL}_{\text{to source}} \;+ \text{TS}$$

Rearrange:

$$\text{TS} = \text{EL} - \text{SL} + 2\ \text{TL}$$ (anything missing?)

Sonar Equation Schematic



Transmit

Receive

Active vs Passive Systems

$$\text{SPL} = \text{SL} - \text{TL}$$

Passive

energy received = energy transmitted * fraction not absorbed * fraction not spread

$$I_r = \underset{\substack{\uparrow \\ \text{source}}}{I_o} \underbrace{\left(\frac{r_o}{r}\right)^2 10^{-\alpha\left(\frac{r-r_o}{10}\right)}}_{\text{transmission loss}}$$

one-way system

Active

$$\text{SPL}_{\text{rec}} = \text{SL} - 2\text{TL} + \text{TS}$$

two-way system

+ noise (Σ thermal, electrical, anthropogenic)

Active vs Passive Applications

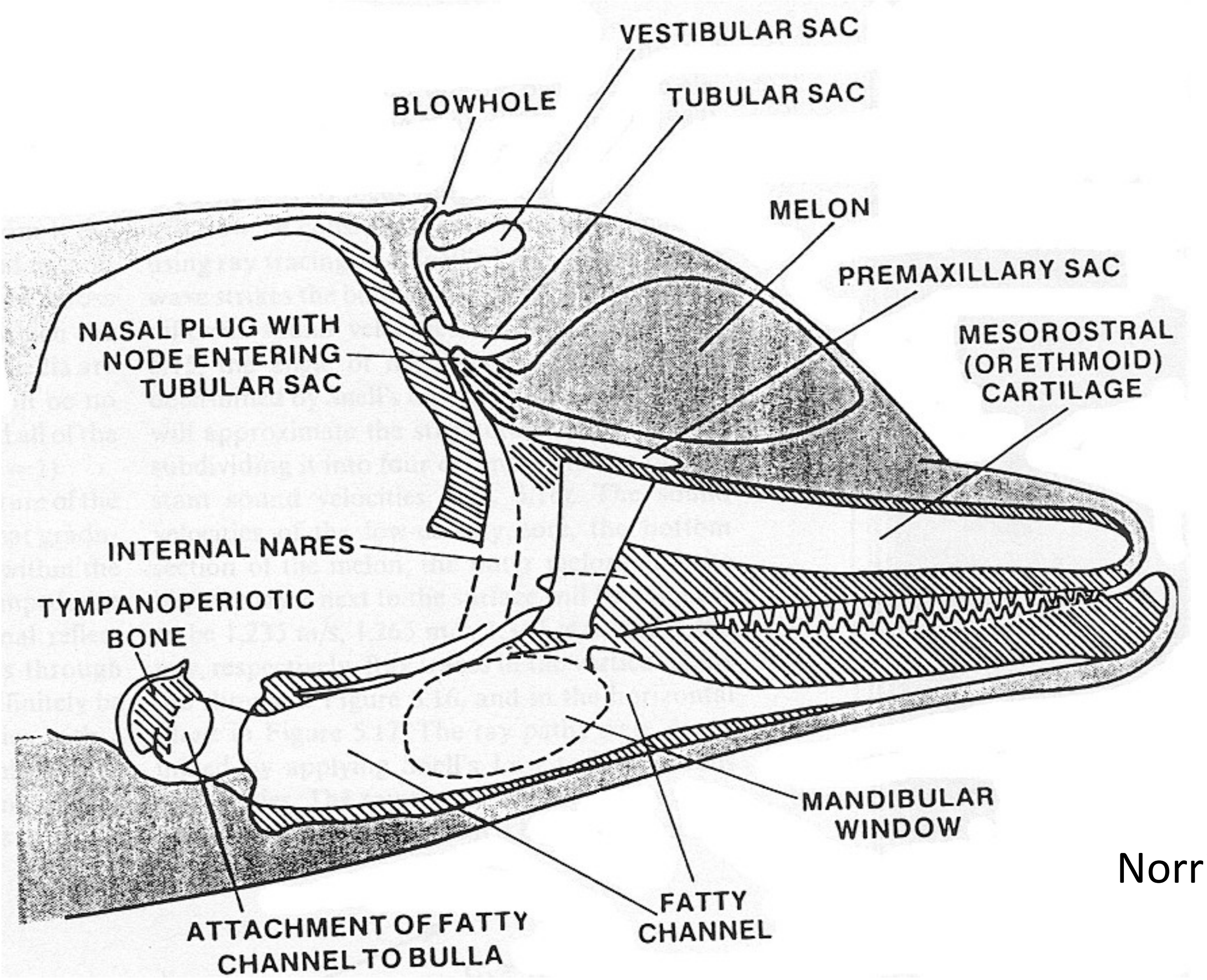
Passive

- determine how far away we can hear animals
- determine how far away animals can hear us (and corresponding intensity level)
- determine range of animal communication
- census populations
- monitor behavior

Active

- determine how much power needed to detect animal
- determine range of predator detecting prey
- census and size populations
- map distributions of animals

Dolphin Sound Reception



Additional Structures:

Pathway of sound to the cochlea is via the lower jaw (cf. Bullock et al. 1968; McCormick et al. 1970)

Norris 1968